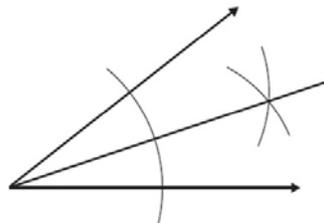


COMPASS AND STRAIGHTEDGE CONSTRUCTIONS

From Basic to Advanced



JUNE 13, 2016

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U of U Summer Teacher's Workshop, Zermatt Resort, Midway, UT

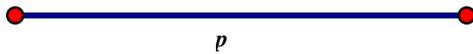
What's in a Number?

When we do Euclidean constructions, we are only allowed to use a compass and a straightedge. We start with two points in the plane (these points can define our unit distance). Then, to construct new points we must follow the three following rules:

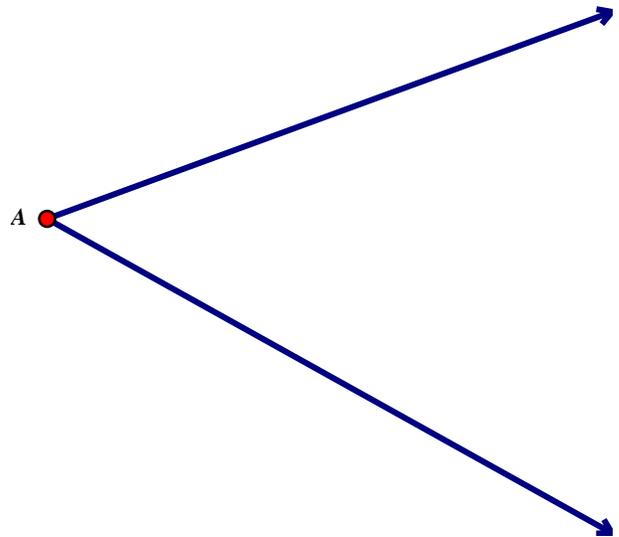
1. Any new point we construct needs to be the intersection of two lines, two circles or a line and a circle.
2. Lines are drawn with the straightedge and must pass through two points we have already constructed.
3. Circles must be centered on points we have already constructed, and their radius must be the distance between two points we have already constructed.

Let's see if we remember a few basic constructions.

1. Given a segment with length p , duplicate (construct a new segment congruent (equal in length) to) the given segment. Construct this new segment on line m with one endpoint at A .

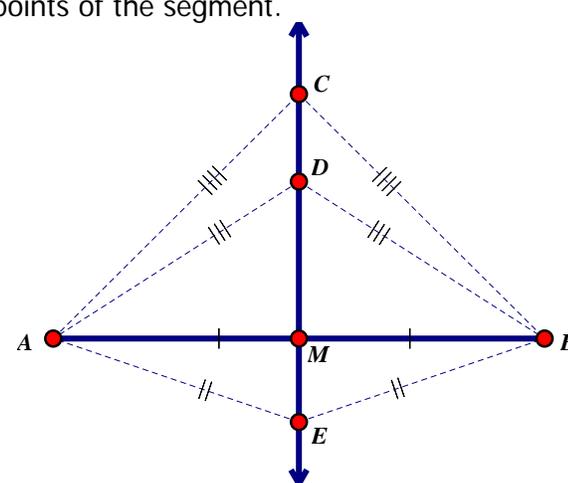
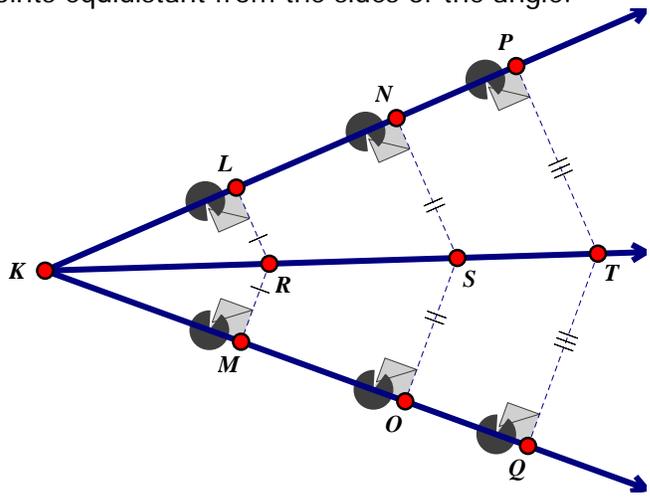


2. Given angle A , duplicate (construct a new angle congruent (equal in measure) to) the given angle. Construct this new angle using ray r as one of its sides.

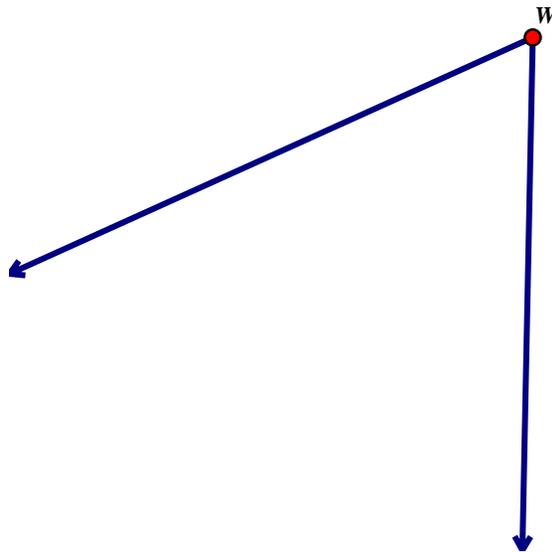
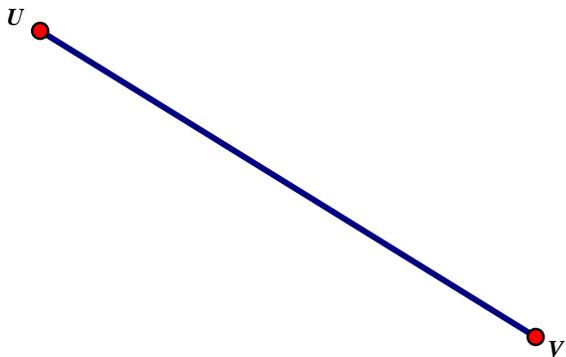


3. Can you give a convincing argument explaining why the constructions above work? Discuss with a neighbor.

We can use a *locus* definition to define certain geometric objects. A locus is a set or collection of points that meet some constraint or condition. For example, a circle is the locus of points equidistant from a given point. The given point is the center of the circle, and the given distance is the radius of the circle.

<p>Definition: The perpendicular bisector of a segment is the locus of points equidistant from the endpoints of the segment.</p>  <p style="text-align: center;">$AC = CB, AD = DB, AM = MB, \text{ and } AE = EB$</p>	<p>Definition: The bisector of an angle is the locus of points equidistant from the sides of the angle.</p>  <p style="text-align: center;">$LR = RM, NS = SO, \text{ and } PT = TQ$</p>
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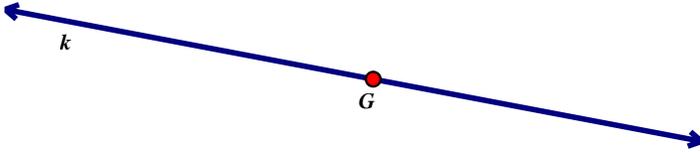
4. Construct the *perpendicular bisector* of segment *UV*. 5. Construct the *bisector* of angle *W*.



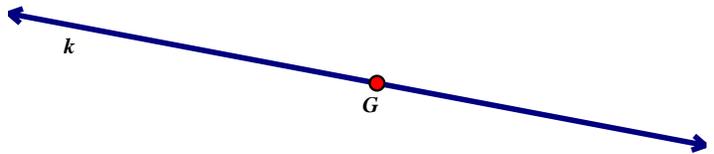
6. Explain why the construction in #3 is the *perpendicular bisector* of segment *UV*. (hint: use the locus definition)
7. Explain why the construction in #4 is the *bisector* of angle *W*. (hint: you may use properties/ symmetries of special quadrilaterals)

Now that we remember how to construct perpendicular bisectors, let's construct *perpendiculars to lines*.

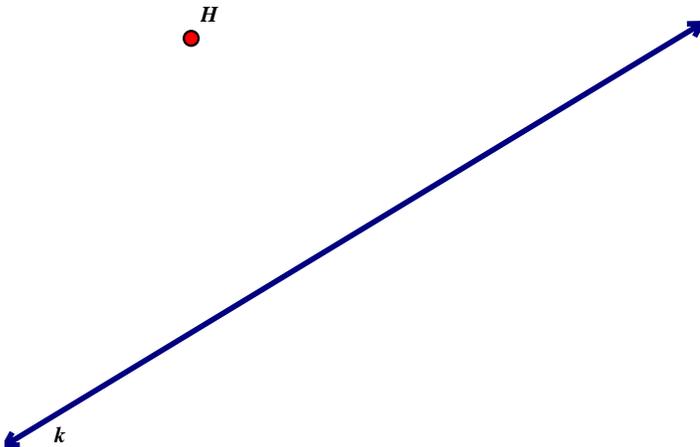
8. Construct the line through point G perpendicular to line k .



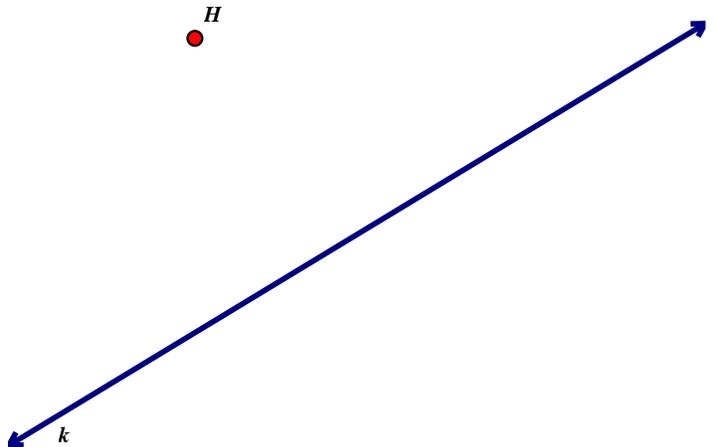
9. Can you do the same construction a different way?



10. Construct the line through point H perpendicular to line k .



11. Can you do the same construction a different way?

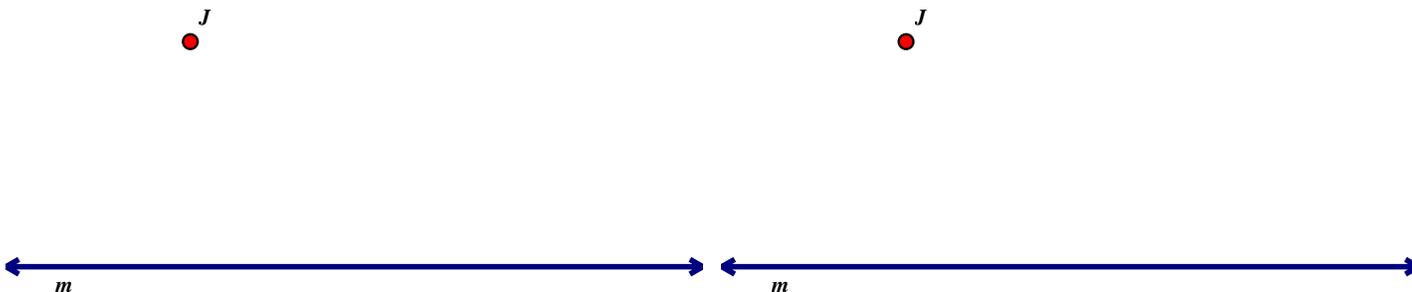


12. Can you give a convincing argument explaining why each of the constructions above work? Discuss with a neighbor.

There is one other basic construction that we are going to need. Do you remember how to construct a line *parallel* to a given line through a point not on that line?

13. Construct the line through point J that is *parallel* to line m .

14. Can you do the same construction a different way?



15. Can you explain why these constructions work? Discuss with a neighbor.

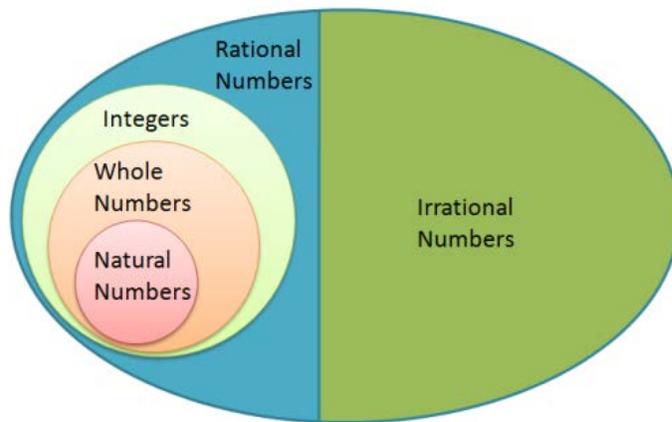
Numbers and Constructability

A constructable number is a number whose length can be constructed from a unit length using only compass and straightedge, following the traditional rules of Euclidean constructions stated above.

Which numbers are constructable? To answer this question, first let's talk about subsets of the complex number system. Traditionally we have divided the set of real numbers into two mutually exclusive sets; the set of rational numbers and the set of irrational numbers, as illustrated in the Venn diagram.

Without too much convincing, I think we can all agree that any natural number can be constructed by simply duplicating the unit segment along a line any number of times. If we consider opposite directions, then we could construct any integer. We could also construct a length representing the sum or difference of two integers, as well as an integer divided by two.

16. Construct the number 3 on line m starting at D .



Use this unit length for exercises 16 and 17

1 unit

17. Construct the number $2\frac{1}{2} = \frac{5}{2}$ on line m .

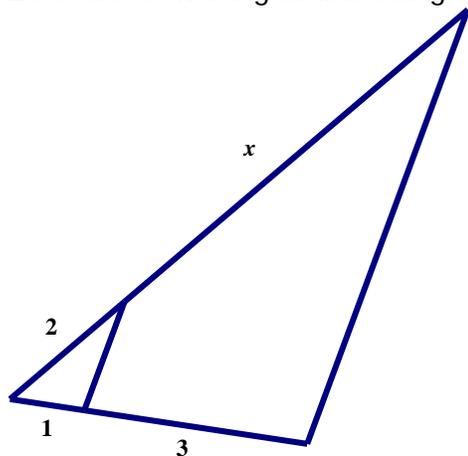


One way to construct an integer that is the product of two integers is to think of multiplication as repeated addition.

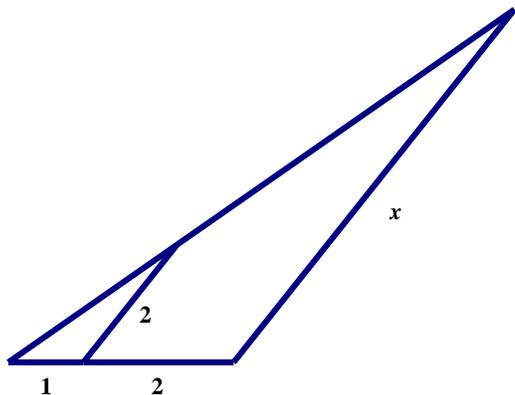
18. Using the unit length $\overline{1 \text{ unit}}$, construct the number that is the product of 2 and 3.



20. Solve for x using similar triangles.

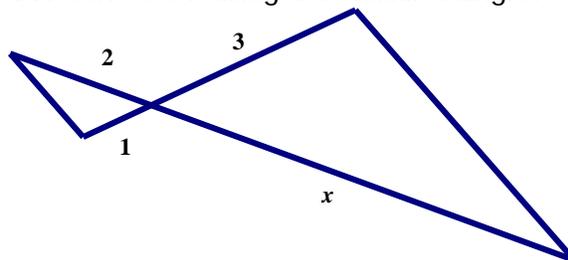


22. Solve for x using similar triangles.

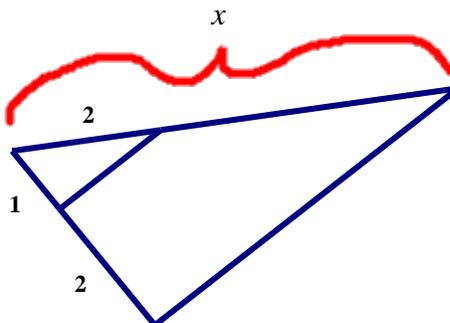


We can also construct the product of two integers by using similar triangles.

19. Solve for x using the similar triangles.



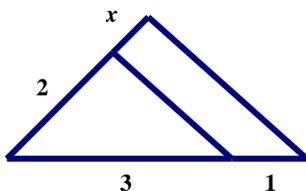
21. Solve for x using similar triangles.



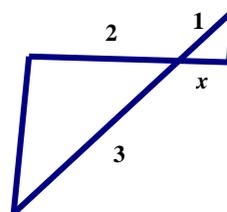
23. Explain how to construct a number that is the product of 2 integers. Why do we need the unit length?

One way to construct a rational number is to think of it as the quotient of two integers, and then use similar triangles as above.

24. Solve for x using similar triangles.



25. Solve for x using similar triangles.



26. Using this unit segment, $\overline{1 \text{ unit}}$, construct a length representing 3 divided by 4.

27. Using this unit segment, $\overline{1 \text{ unit}}$, construct a length representing the rational number $\frac{4}{3}$.

28. Divide line segment JK into four congruent parts.

29. Trisect segment AB .



30. Divide segment EF into five congruent parts.

31. Explain how you would divide a line segment into any number of equal parts.



We have now covered how to construct any *rational number*, and the four basic operations on rational numbers; addition, subtraction, multiplication, and division. But what about *irrational numbers*? Can we construct any irrational number?

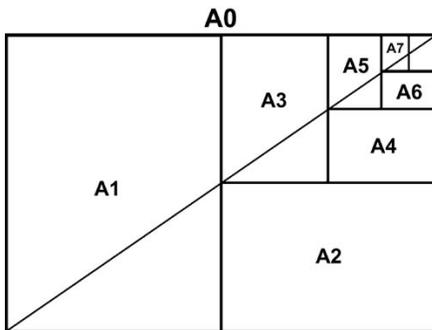
Let's start by looking at the A-series paper sizes. The A-series paper dimensions are pretty much standard everywhere in the world except for the United States and Canada.

The A-series paper starts out with size A0, which is defined to have an area of 1 m^2 . In addition, A-series paper has the property that if you fold it in half, it maintains the same aspect ratio (the two smaller rectangles are similar to the larger rectangle).



32. Using the folding property described above, find the aspect ratio of all A-series paper.

33. Using the aspect ratio you just found, and the fact that A0 paper has an area of 1 m^2 , find the exact dimensions of A0 paper and compare them with the dimensions listed in the table below (which are rounded to the nearest mm).



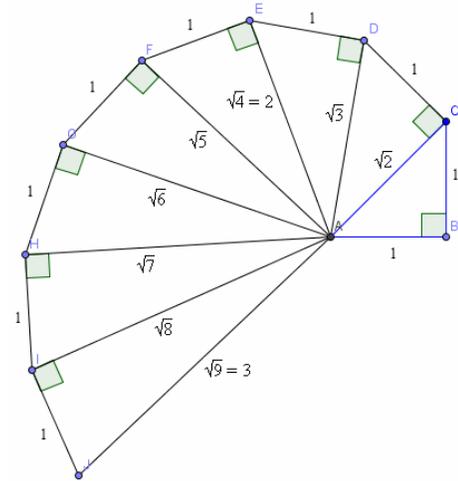
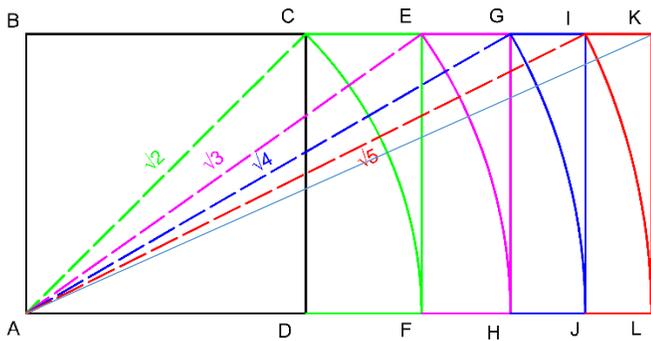
34. Construct this aspect ratio using the given unit segment.



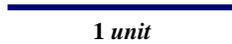
Paper Size	Length (mm)	Width (mm)	Area (m^2)
A0	1189	841	1.0
A1	841	595	$1/2$
A2	595	420	$1/4$
A3	420	297	$1/8$
A4	297	210	$1/16$
A5	210	148	$1/32$
A6	148	105	$1/64$
A7	105	74	$1/128$
A8	74	53	$1/256$

35. Construct a rectangle in the dimensions of A-series paper using the given unit segment as the short side.

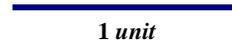
Repeated use of the Pythagorean Theorem will allow us to construct the square root of any natural number as demonstrated in the two different diagrams.



36. Using the method shown above, construct the irrational number $\sqrt{3}$ using the given unit segment.



37. Using the method shown above, construct the irrational number $\sqrt{3}$ using the given unit segment.

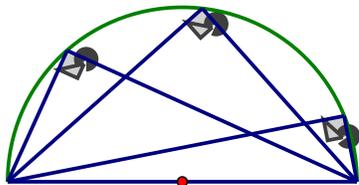


38. Using the given unit segment above, construct $\sqrt{\frac{3}{4}}$
(hint: rationalize the denominator)

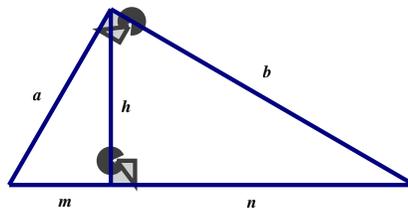
39. Using the given unit segment above, construct $\sqrt{\frac{2}{3}}$
(hint: rationalize the denominator)

The solutions to problems **38** and **39** hint at a procedure to construct the square root of any rational number. Although in theory it seems straight forward to do these constructions, in practice it sometimes gets tedious carrying out the steps of constructing perpendiculars and parallel lines, etc., in order to arrive at our final goal.

One technique I have found helpful is an alternative method for constructing square roots. This method relies on two geometric relationships. First, any angle inscribed in a semicircle is a right angle, and second, the altitude drawn to the hypotenuse of a right triangle divides the triangle into two smaller triangles that are similar to each other and to the original triangle.



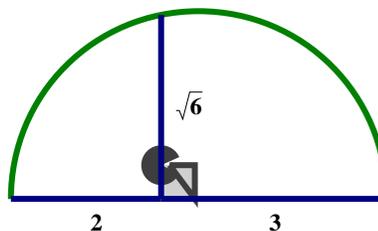
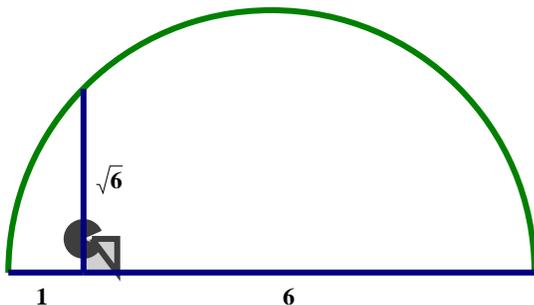
The 3 inscribed angles are right angles.



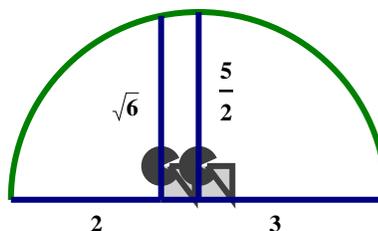
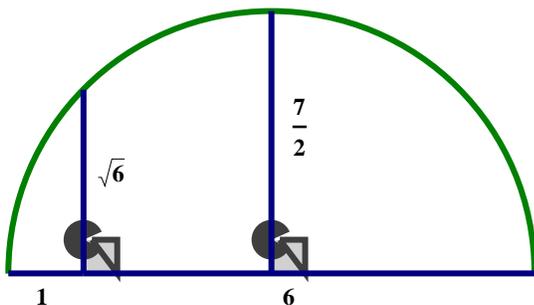
$$\frac{m}{h} = \frac{h}{n} \quad \text{or} \quad h^2 = mn \quad \text{or} \quad h = \sqrt{mn}$$

Using similar triangles we can conclude that the length of the altitude to the hypotenuse of a right triangle is the geometric mean of the two segments of the hypotenuse (in addition, each leg is the geometric mean of the segment of the hypotenuse adjacent to that leg, and the entire hypotenuse). Inscribing an angle in a semicircle gives us an easy way to construct a right (tri)angle.

Here are two different ways to construct $\sqrt{6}$ using this technique. In words, the square root of six is the geometric mean of one and six, or two and three.



In problem **17** we constructed the rational number $\frac{5}{2}$ which can be thought of as the arithmetic mean of 2 and 3, ie. $\frac{2+3}{2}$. Now that we know how to construct the geometric mean of 2 and 3, we can visually see why the geometric mean of two numbers is always less than or equal to the arithmetic mean of those two numbers.



If we wanted to construct $\sqrt{6}$ using the sequence of right triangles by continually adding a new leg of length one, we wouldn't have to construct all five previous triangles to get a hypotenuse of $\sqrt{6}$. We could just start out with the triangle whose longest leg is the biggest perfect square less than six and then continue from

there, ie., $\sqrt{2^2 + 1^2} = \sqrt{5}$, then $\sqrt{(\sqrt{5})^5 + 1^2} = \sqrt{6}$. But it is much easier to construct $\sqrt{6}$ using geometric means.

Given the unit segment $\overline{\hspace{1cm}}$ *1 unit* can you construct $\sqrt{12}$ four different ways using the geometric mean method?
40. **41.**

42.

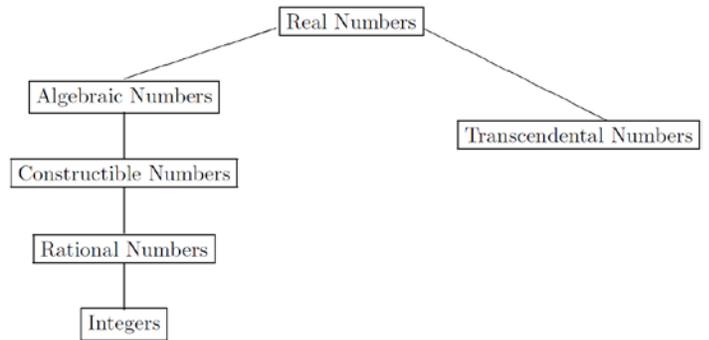
43.

Back to our question, which *irrational* numbers can be constructed? We have seen that we can construct the square root of any rational number. We can also construct the sum, difference, product, and quotient of a rational number with a square root of a rational number, or the sum, difference, product, and quotient of two or more square roots of rational numbers.

In order to fully answer this question, let's look at another way of dividing the set of real numbers into subsets. The real numbers can be divided into two mutually exclusive sets, the *algebraic numbers* and the *transcendental numbers*, as illustrated in the diagram to the left. An *algebraic number* is any number x that satisfies some algebraic (polynomial) equation of the form

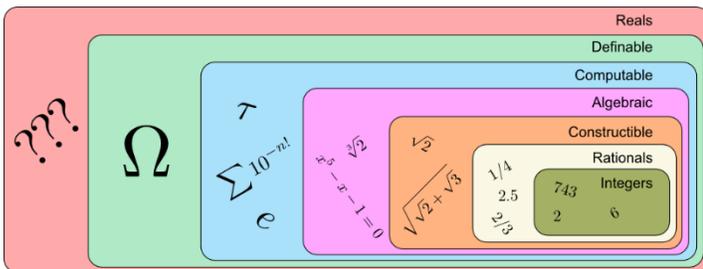
$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x^1 + a_0 x^0 = 0$$

where $n \geq 1, a_n \neq 0$, and the a_k are integers.

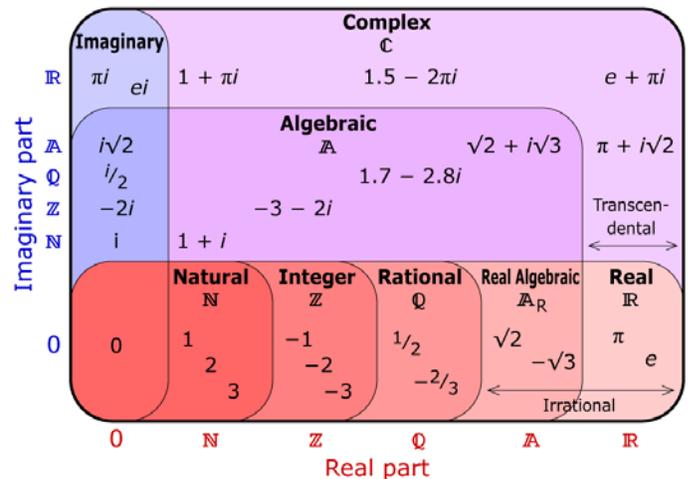


For example, $\sqrt{2}$ is an *algebraic number*, since it satisfies the equation $x^2 - 2 = 0$, and we say that the *algebraic number* $\sqrt{2}$ is of degree 2 since it is a solution of a 2nd degree polynomial equation. The rational number $\frac{2}{3}$ is an *algebraic number* of degree 1 since it is a solution of the 1st degree equation $3x - 2 = 0$. And $2\sqrt[3]{2}$ is an *algebraic number* of degree 3 since it is a solution of the 3rd degree equation $x^3 - 16 = 0$. Not all irrational numbers are algebraic. For example, π is not *algebraic*, since it is not the solution of a polynomial equation with integer coefficients. We say that π is a *transcendental number*, because it "transcends" the properties of *algebraic numbers*. *Transcendental numbers* cannot be constructed with compass and straightedge.

As can be seen in the diagram, all *constructible numbers* are *algebraic*, but not all *algebraic numbers* are *constructible*. A *constructible number* can be represented by a finite number of additions, subtractions, multiplications, divisions, and square root extractions of integers, (but no extractions of higher order roots).



If a cubic equation with rational coefficients has no rational root, then none of its roots is *constructible*. A complex number is *constructible* if the point representing its location in the complex plane has coefficients which are *constructible*.



Because constructible numbers are algebraic, we can find the polynomial equation of which it is a solution. These polynomial equations will always have a degree which is a power of two. For example:

$$\begin{aligned} x &= 1 + \sqrt{3} \\ x - 1 &= \sqrt{3} \\ x^2 - 2x + 1 &= 3 \\ x^2 - 2x - 2 &= 0 \end{aligned}$$

$$\begin{aligned} x &= \sqrt{1 + \sqrt{3}} \\ x^2 &= 1 + \sqrt{3} \\ x^2 - 1 &= \sqrt{3} \\ x^4 - 2x^2 + 1 &= 3 \\ x^4 - 2x^2 - 2 &= 0 \end{aligned}$$

44. Construct the irrational number $x = \sqrt{2} + \sqrt{3}$

1 unit

45. Construct the irrational number $x = \sqrt{5 + 2\sqrt{6}}$

1 unit

46. Find the polynomial equation for $x = \sqrt{2} + \sqrt{3}$

47. Find the polynomial equation for $x = \sqrt{5 + 2\sqrt{6}}$

48. Construct the irrational number $x = \frac{1+\sqrt{5}}{2}$

1 unit

49. Find the polynomial equation for $x = \frac{1+\sqrt{5}}{2}$

Now that we have explored compass and straightedge constructions with the classical tools and are starting to feel comfortable with them, let's revisit some of these same ideas using modern tools. Doing constructions by hand is important to help us slow down and think about what we are doing, but the tediousness and errors in precision can be frustrating at times, and all of our constructions are static and only show us that one case we are working on.

Behold the power of visualization! If you don't have [GeoGebra](http://www.geogebra.org/) already, visit <http://www.geogebra.org/> to download it. It is free! You can run it with a web browser or as its own application on your computer, tablet, or android (yes, hours of fun and entertainment on your phone while waiting at the airport).

I suggest you duplicate the following constructions, and explore the new ones listed. After each construction, click and drag some of the points to see what happens. If you don't know how to do a construction using the features of the software, or know the terminology, use the "help" feature, ask a neighbor, or "google" it.

1. Draw a segment and duplicate it.
2. Draw an angle and duplicate it.
3. Construct the perpendicular bisector of a segment (there might be a tool for this). Create an arbitrary point on the perpendicular bisector and measure from it to each endpoint of the segment. Drag the point and the endpoints of the segment.
4. Construct the bisector of an angle. Create an arbitrary point on the angle bisector and construct the line through this point perpendicular to one of the sides (using the perpendicular line tool). Construct the circle centered around the point on the angle bisector, and through the point that is the foot of the perpendicular line. Click and drag the center of the circle.
5. Use the polygon tool to draw a triangle. Construct a line through one vertex that is perpendicular to the opposite side. Construct a new triangle with one vertex on this parallel line, and the other two vertices as the vertices or the opposite side of the original triangle. Measure the areas of both triangles and explore.
6. Use the "segment with a given length" tool to construct a unit length. Use the segment tool to create two arbitrary segments. Construct the segment that represents the product of the two segments. Construct the segment that represents the quotient of the two segments. Construct the arithmetic mean of these two segments (you don't need to use the unit segment). Construct the geometric mean of these two segments (again, the unit segment is not necessary).
7. Divide a segment into 5 equal parts.
8. Construct the tangent to a circle through a point on the circle.
9. Construct the tangent to a circle through a point outside the circle.
10. Construct the circumcenter of a triangle and circumscribe a circle around the triangle. Explore.
11. Construct the centroid of a triangle. The centroid divides each median into two parts. Measure the length of each part and make a conjecture about the relationship between these two lengths.
12. Construct the orthocenter of a triangle. The set of 4 points consisting of the vertices of a triangle and its orthocenter are called an orthocentric set. What does this mean?
13. Construct the incenter of a triangle. Inscribe a circle in a triangle.
14. Construct the circumcenter, centroid, orthocenter, and incenter all on the same triangle (it might help to hide some constructed objects once you have used them). Drag a vertex of the triangle. Do you notice anything special about these points?

In the view menu, select "3D Graphics" to open a new window. Right-click and drag to change the orientation of the visual plane and coordinate axis.

1. Using the "pyramid" tool, construct a triangular pyramid (tetrahedron) with its base in the visual plane and the vertex opposite the base above the visual plane. Explore by dragging the different vertices and right-clicking and dragging to change the orientation of the view.
2. Can you construct the circumcenter of the tetrahedron and circumscribe a sphere around the tetrahedron? (hint: what would the locus of points in space equidistant from the endpoints of a segment look like?)

Other geometry software I recommend:

-  **Cinderella**. A **free** dynamic geometry construction program with Euclidean, Hyperbolic, and Spherical environments. <http://cinderella.de/tiki-index.php>
-  **Poly** \$28. A program that renders various polyhedra that can be rotated and viewed from different angles. You can view a wire frame of the object, along with unfolding it to see its 2D net. You can also view a dynamic Schlegel diagram of each polyhedron. <http://www.peda.com/>
-  **Cabri 3D** 75 Euros. I learned 3D geometry on this software, so it is dear to my heart, but most everything and more can be done now in the 3D graphics view of Geogebra. <http://www.cabri.com/cabri-3d.html>
-  **The Geometer's Sketchpad** \$70. Again, I grew up on this software, but Geogebra can do most everything and more that Sketchpad does. <http://info.mheducation.com/sketchpad.trial.html>
-  **Microsoft Mathematics**. A **free** add-in for Microsoft Office products or a stand-alone program that has a graphing calculator and other tools. I especially like the 3D equation graphing. <https://www.microsoft.com/en-us/download/details.aspx?id=15702>

Other products and ideas:

- **Lénárt Sphere**, \$80. A physical model with tools to do spherical geometry constructions and investigations. <http://lenartsphere.com/>
- 3D printing
- **Hoberman Sphere**, \$35. A toy made of linkages that expands. I like to use this to demonstrate related rates between radius, surface area, and volume. <https://www.amazon.com/Hoberman-HS104-Original-Sphere-Rainbow/dp/B00000JN49>
- Other ideas?